Explaining Mistakes in Single Digit Multiplication: A Cognitive Model

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Abstract

Error patterns for arithmetic problems are very rich in information, but they are hard to investigate systematically because of the small number of mistakes made. To be able to investigate errors in arithmetic we therefore used an online educational application called Math Garden, which teaches children arithmetic in the form of several different tasks. Because of the large number of users, Math Garden provides sufficient data to investigate errors systematically. Using the Math Garden data set, we developed a cognitive model in the PRIMs architecture that can give a comprehensible account of the errors made in single-digit multiplication problems. The model does a relatively good job of explaining errors on easy problems, but has difficulties explaining mistakes for harder problems. In addition to the current model, we propose some approaches to improve the model to explain mistakes in the harder problems as well.

Keywords: Arithmetic, Multiplication, PRIMs, Errors

Introduction

Arithmetic is one of the core skills taught in primary education. Especially single digit multiplication is considered to be one of the core abilities in today's world. This kind of basic skill is hard to study in adults, because it is so well trained and automatic that hardly any mistakes are made. Even children make few errors. This is unfortunate, because errors give great insight in the processes and especially the strategies underlying these skills.

In this paper, we will investigate the different kinds of errors children make in multiplication by employing a huge data set from a web-based practice program, Math Garden (e.g. Klinkenberg, Straatemeier, & Van Der Maas, 2011). Math Garden is used by thousands of students every day, and the data therefore include a significant amount of errors. To explain the errors children make, we will develop a comprehensible cognitive model of these error data. Our goal is to gain insight into the processes and strategies underlying single-digit multiplication.

Existing Models of Multiplication

Previously, several models have been built to explain patterns found in arithmetic data. We can discern three categories in this regard: memory strength models, network interference models, and computational efficiency models (Ashcraft & Guillaume, 2009). Memory strength models and network interference models both put the emphasis on memory retrievals, while strategy-based models implement the use of algorithmic strategies, such as repeated addition – e.g. solving 8 + 8 + 8 instead of 3 x 8 – and counting

in steps of two, three, or more – e.g. 3, 6, 9 to solve 3 x 3. Most models combine a strategy-based approach with a memory-based approach: retrieval based models often include some kind of computational strategy, and strategy based models often also include a rehearsal strategy (e.g. Lebiere, 1999; Siegler, 1988).

An example of a combined model is the model by Siegler (1988). The main strategy in this model was retrieval, which was tried multiple times. Every trial a random number of retrievals was attempted to find an answer to the problem. In the model's declarative memory, each exercise was not only connected to the correct answer, but also to incorrect answers. The more problem-answer connections there are, the harder it is to retrieve a correct answer.

Because of the associations with incorrect answers, retrievals could also result in an incorrect answer. Therefore, the associative strength of each successful retrieval was compared to a confidence criterion that was set at random in each trial. If the associative strength of the retrieved answer was lower than the confidence criterion, the answer was rejected and a new retrieval was started. Only when everything else failed, an alternative strategy would be applied. This alternative strategy was an algorithmic process, such as repeated addition, in which one multiplicand is added the number of times of the other multiplicand.

According to Siegler, the strategies that are initially used to solve the problem determine which problem-answer connections end up in the declarative memory. In turn, the problem-answer combinations in memory influence the strategies that are used to solve the problem. Thus, the errors children make early on in their development of the

Figure 1. The multiplication task in Math Garden. Responses are given using the keypad. On the bottom of the screen the current value of the problem is represented with coins.

multiplication skill are of great importance for the further development of the strategies they use.

In 1999, Lebiere implemented a combined memory and computational strategy model within the constraints of the cognitive architecture ACT-R. This model, similar to Siegler (1988), tries to retrieve the answer to an arithmetic problem, and, if that fails, uses repeated addition to calculate the answer. In the model by Lebiere, three sources of errors can be discerned. The first source of error stems from the use of repeated addition. For example, when solving the problem `3 x 8`, a student can accidentally do one additional step, resulting in `32`, or one step too few, resulting in `16`. Alternatively, an addition mistake in one of the steps can result in an answer such as `16 + 8 = 25`.

When retrieval becomes the main strategy to solve these problems, the errors made in repeated addition will still have an influence in the form of incorrectly stored combinations of a problem and its response. These lingering associations are the second source of error. The third source of error in the model is that of a partial match between stored information and to be retrieved information. For example, when the problem `3 x 5` needs to be retrieved, a mistake can be made by retrieving a similar problem, such as `3 x 4 = 12`, and giving the answer to that problem instead.

Current Study

Both Siegler and Lebiere did analyze the errors in their data. However, because of the relatively small datasets they used to fit their models, it was hard to examine the error patterns in a more systematical way:

> *"The patterns of errors for multiplication are quite rich, but harder to examine systematically because they take place over a wider range of values and display some characteristics (table errors, close misses, etc.), which are difficult to average over and plot together. For those reasons, let us concentrate on the pattern of errors for a single problem"* (Lebiere, 1999)

We will continue the work of Lebiere by looking in more detail at the errors in a large dataset that gathers the data of thousands of primary school students in all age categories. Although the proportion of errors is still low, the sheer amount of data makes the systematical analysis of these errors possible. Using these data, we can make finer grained assumptions about the strategies underlying these mistakes.

In this paper, we will start this endeavor by looking at three specific problems. First we will discuss the data and the different mistakes we find in the data, then we will propose strategies that can have led to these errors. We will implement these strategies in the cognitive architecture PRIMs (Taatgen, 2013) and discuss the similarities and differences between the results of the model and the data.

Task & Data

The data were gathered from "Math Garden" (see also: van der Ven, Straatemeier, Jansen, Klinkenberg, & van der Maas, 2015), an online computer application that is used by school children in the Netherlands to practice math and arithmetic. It offers problems that are adapted to the capabilities of the user, so that each problem has a reasonable chance of being solved (the default probability of correctly solving a problem is .75, this can be adjusted by the user). While the program contains a wide variety of different tasks, we will focus here on a standard multiplication task (see Figure 1).

Participants

Because we only have access to aggregated data from Math Garden, the specific distribution of participants is unknown. The users of Math Garden are Dutch primary school children with ages roughly between 5 and 13 years.

The Multiplication Task

The task we focus on in this paper is the multiplication task. In this task, a multiplication problem is presented on the screen (Figure 1). The student has to solve this problem within 20s. The answer is given by clicking on an on-screen keypad. Time is represented as a row of coins and every second a coin disappears from the screen. The coins that are left on the screen when the student has entered the answer are the score that is received, in the case of a correct answer, or lost, in the case of an incorrect answer. No points are awarded or lost when no answer is provided. This way of scoring is known as the 'High Speed High Stakes' principle (Maris & van der Maas, 2012). Students can decide for themselves how many trials they want to play and when they want to play.

Data

The data set we used was obtained on 25 May 2015 and contains the data of 8,489,703 attempts of 81 different problems (`1 x 1 – 9 x 9`). The overall percentage of errors in the full dataset was 10.42%. This is lower than the expected error percentage of 25% because late answers and answers in which the student asked for a hint were not taken into account. We will give a qualitative overview of the types of errors children make.

The most common errors often fit in one of the following categories:
1. The student has added the numbers instead of multiplying them. For example, `3 x 2 = 5`.
2. Operand related mistakes: the answer is consistent with the answer to a very similar problem. For example, `3 x 4 = 15`, which is the answer to the problem `3 x 5`, or `6 x 7 = 35,` which is the answer to the problem `5 x 7`.
3. Miss 1 errors: the answer is very close to the correct answer. For example, `3 x 5 = 14`, which is the

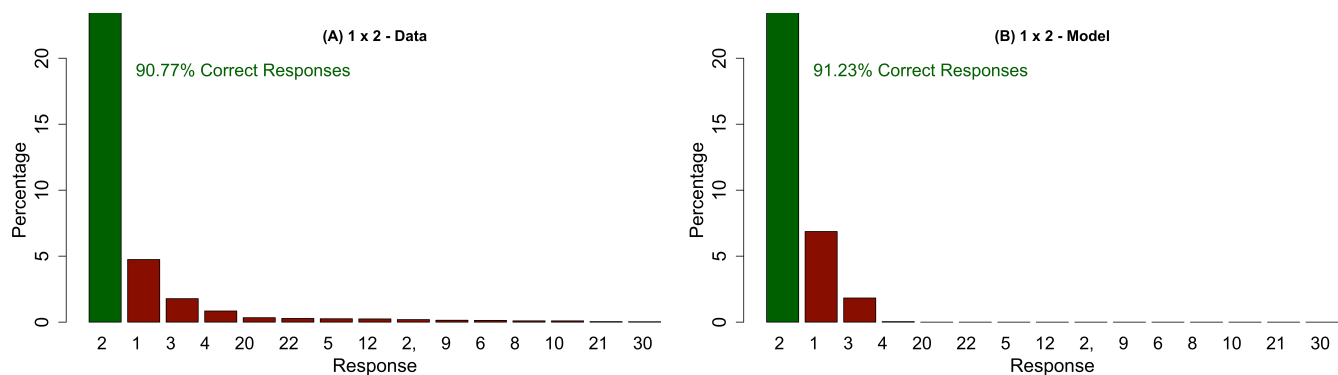Figure 2. Responses given in the data (A) and by the model (B) after a simulation of 25 times 400 trials of the problem `1 x 2`.

correct answer minus one, `or 6 x 7 = 43`, which is the correct answer plus one.

Most of these mistakes can be explained by a mistake in the calculation or retrieval procedure. Interestingly, there are also mistakes that cannot easily be explained in the same manner. While previous models focused mainly on the first types of error, the goal of our model is to also explain some of the other mistakes. For ease of exposition we will focus on the errors in three different problems: `1 x 2`, `3 x 4`, and `9 x 6`. These problems are chosen because they fall in the first, second, and third tertile of the data, based on the Math Garden estimate of the average level of the students who have solved the problem correctly.

Explaining Multiplication Mistakes

We will discuss the five most commonly made mistakes for the abovementioned problems and how they are implemented in the model. Reading errors and input errors are outside the scope of the current model. We will start out by discussing the mistakes (see Figures 2A-4A) and hypotheses for the origin of these mistakes. Afterwards we will explain the setup of the model, and what the model can or cannot explain.

1 x 2

The problem `1 x 2` is an easy problem and was recorded 60,821 times in our dataset. In 91% of the cases the problem was solved, in the other 9% errors were made. The most common mistakes for the problem `1 x 2` are shown in Figure 2A.

The most common mistake is to give the answer 1, which is a pattern of behavior seen in all problems where one of the multiplicands is a 1. There are several possible explanations for this mistake: (1) It could be caused by partial matching, instead of retrieving the required result for `1 x 2`, the result for `1 x 1` is retrieved. However, given that we also observe this pattern for `1 x 9`, this is unlikely. (2) The rule for multiplication by 0 can be overgeneralized. When a number is multiplied by 0, the answer is also 0. Since the tables for 0 and 1 or often the first multiplication tables a student encounters, they might use the rule they have learned and used successfully for the table of 0 in the

table of 1. (3) Finally, this mistake could be due to an incorrect application of the 1-rule. For all problems of the form `1 x N` it holds that the answer is `N`. If this rule is remembered or applied incorrectly, the result may be that the other multiplicand, the 1, is considered to be the correct response.

The second, third, and fourth mistakes are 3, 4, and 20. All of these answers fall into one of the categories mentioned earlier. 3 corresponds to either the addition of both multiplicands, an answer that corresponds to a similar problem, or miss 1 error. 4 corresponds to an operand related mistake, namely `2 x 2 = 4`. The most likely explanation for 20 is that the 1 is mistaken for a 10. This confusion is the third most made mistake in the table of 1, but it does not happen in any of the other tables. A likely explanation for the confusion is that the multiplication table for 10 is taught as one of the first multiplication tables and is therefore well known. The 1 is then easily confused with the 10.

3 x 4

The problem `3 x 4` is a medium level problem; it was recorded 148,279 times in our dataset. 90% of the problems where solved correctly. In 10% of the cases an error was made. A specification of the errors can be found in Figure 3A.

The main mistake found in the data is 16. 16 is the answer to `4 x 4`, an operand related error. It fits with the most common error categories we described before. It can either be caused by a retrieval that has gone wrong, either due to a previous mistake or due to a partial matching error. The other possibility is that the mistake is made because of a mistake in repeated addition; the student took one step too many in the calculation of the answer.

The next most common mistake is 9. 9 is the answer to `3 x 3`, also an operand related error. Therefore, a similar explanation to the previous mistake applies.

The final three mistakes we will discuss here – 8, 7, and 15 – are made less often, but can be explained in a similar way. 8 is the answer to `2 x 4`, 7 is the answer to `3 + 4`, and 15 is the answer to `3 x 5`. In all these cases, this is
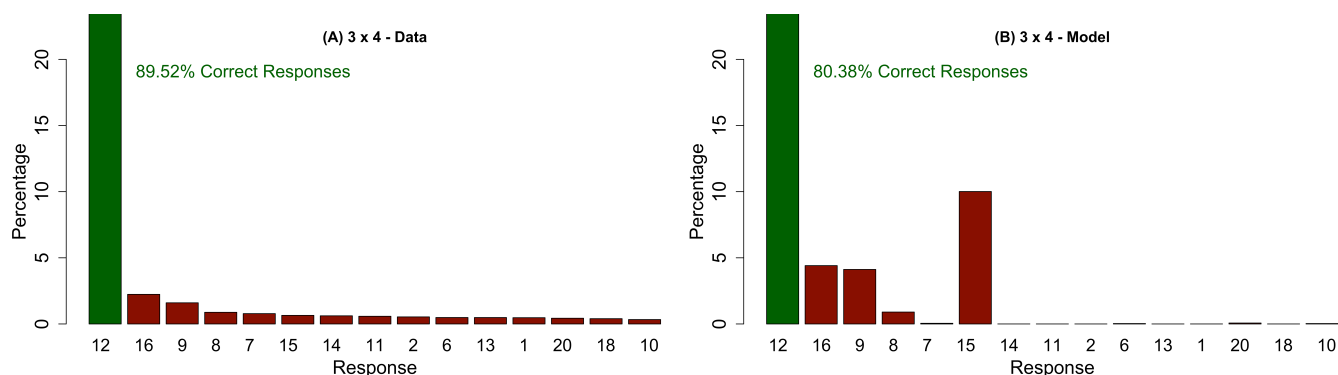
Figure 3. Responses given in the data (A) and by the model (B) after a simulation of 25 times 400 trials of the problem `3 x 4`.

either a mistake in the retrieval, a mistake in the repeated addition procedure, or a wrongly applied procedure (in the case of `3 + 4 = 7`).

6 x 9

`6 x 9` is one of the more difficult multiplication problems. It was recorded 90,145 times. 88% of the problems were solved correctly. The most common mistakes for `6 x 9`, as shown in Figure 4A, are `45`, `56`, `63`, `36`, and `53`. Of these responses, `45` and `63` are one step earlier and one step later in the table of `9`. In contrast, earlier and later steps in the table of `6` do not show up in the most common answers. This could indicate that students learn the commutative property relatively early, and apply repeated addition to the largest multiplicand. This takes fewer steps and therefore there is less opportunity for errors. Other evidence in the early learning of the commutative property comes from the mistakes made for problems and there exact opposite, such as `6 x 9` and `9 x 6`. In nearly all cases, the mistakes made for these two forms of a problem are exactly the same, and made with approximately the same frequency. Since the commutative property also holds for addition problems, it might be that the concept is learned early on for addition problems and transferred to multiplication problems.

`56` is the second most common mistake. This is most likely a mistake that is made because of errors in repeated addition or because the numbers `54` and `56` are very similar. `56` is also the correct answer to another difficult problem, `7 x 8`.

The fourth most common mistake, `36`, is either an input error for the response `63`, or the answer to `6 x 6`. Finally, `53` is probably an addition mistake. Independent of where in the sequence the mistake is made, it is very easy to arrive at a number close to `54`.

Overall, the mistakes made on the problem `6 x 9` seem to be less related to retrieval, and more representative of an algorithmic strategy, such as repeated addition.

A PRIMs Model of Single-Digit Multiplication

Our PRIMs model for multiplication was inspired by Lebiere (1999). The model has rules to retrieve the answer but also to compute the answer using repeated addition. The current model does not give an exhaustive fit of the data, but attempts to explain the most common cognitive mistakes. Errors in reading or input are outside of the scope of this paper. Model results are shown in Figure 2-4B.

PRIMS

PRIMs (Taatgen, 2013) is short for PRimitive Information processing eleMents. It is based on the ACT-R cognitive architecture (Anderson, 2007), but expands it by considering knowledge of tasks in a broader context. For our purposes this means that if PRIMs lacks task-specific knowledge, for example when it does not know a multiplication fact, it may try to determine the answer based on other skills it has, possibly, or even likely, producing an error.

PRIMs uses operators instead of production rules. The main difference is that PRIMs can use its operators for other tasks. This can be beneficial if other operators fill a knowledge gap, but it can also result in an error. Because of this, a model build in the PRIMs architecture does not only resemble someone who is already a perfect problem solver, but it can also account for the initial learning process.

Since PRIMs is based on the ACT-R cognitive architecture, it incorporates many of the mechanisms from this architecture. The declarative memory in PRIMs is based on the declarative memory in ACT-R: it contains chunks with related information. When a retrieval request is send to the declarative memory, the activation of each chunk is determined by the number of previous encounters with that chunk, and the time since those encounters.

The declarative memory of the current model also makes use of a mechanism called partial matching. In partial matching, chunks that do not completely match a retrieval query can also be retrieved. The activation of these chunks gets a mismatch penalty that is conversely proportional with the similarity to the requested chunk.

In PRIMs, the mismatch between numbers is calculated by taking the ratio of the smaller number to the larger number, minus one (based on Lebiere, 1999, p.48).
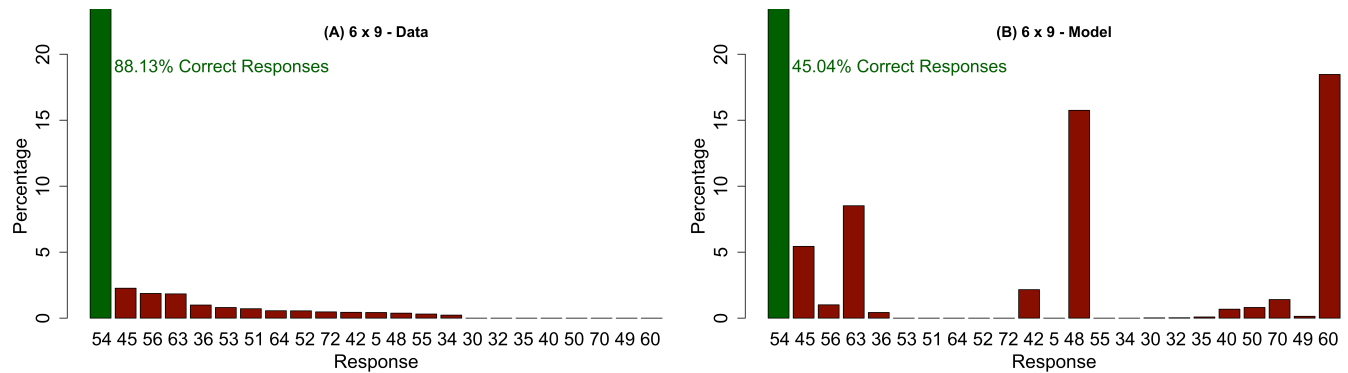
Figure 4. Responses given in the data (A) and by the model (B) after a simulation of 25 times 400 trials of the problem `6 x 9`.

The Model

For the current model, we have chosen to model the problems `1 x 2`, `3 x 4`, and `6 x 9`. For each problem, we assume that the model does not know the current problem at the start of the simulation, but it does know all addition facts and multiplication facts up to the current problem plus one. So for the problem `3 x 4`, all multiplication facts up till `4 x 5` are present in memory. This setup was chosen to simulate a large group of students that each know different problems.

The model in its current form uses standard parameter settings for ACT-R and PRIMs, to achieve these results, no parameter fitting was necessary.

The model first tries to retrieve the answer to the problem from memory; if retrieval fails the answer is calculated using repeated addition. However, a model that is purely based on retrieval and repeated addition cannot explain all the error patterns in the data. For example, the answer 1 is the predominant mistake made in the table of 1, but cannot be explained by a combination of retrieval and repeated addition.

1 x 2

The responses to the problem `1 x 2` are largely explained by the partial matching mechanism. The most common mistake however, `1`, is a mistake in the multiplication table 1, and it is likely that this mistake represents a wrongly applied strategy for solving problems of the form `1 x N` or `N x 1` – as explained above.

To explain this mistake, our model contains a specific strategy that can account for this mistake. This strategy represents the incorrect application of the rule for problems with the multiplicand `1`. When a problem has the multiplicand `1`, the other multiplicand is the answer to the problem. However, when this rule is incorrectly applied, it can easily result in the answer `1` being given, as we have seen in the data. This strategy is implemented in our model by a competition between production rules. When a problem in the form of `1 x N` is encountered, one of two rules can fire. Either the correct rule, which gives the correct response

`N`, or the wrong rule which gives the incorrect response `1`. Over time the model will learn the correct rule through utility learning.

The results for the model on the `1 x 2` problem can be seen in Figure 2B. While this model captures the mistakes for the problem `1 x 2` relatively well, it still has some problems in fitting the error data for the larger problems, `3 x 4` and `6 x 9`.

3 x 4

The most common mistakes to the problem `3 x 4` can for the most part be explained by the partial matching mechanism. However, there is one error that stands out: `15`. We will explain the model's bias for this answer below.

Because the current model has only been applied to instance of specific problems, it does not take into account the frequency of exposure to previous problems. While it is known that problems with smaller multiplicands are practiced more often, this is not represented in our model. In other words, the current model does not incorporate the problem-size effect (e.g. Domahs, Delazer, & Nuerk, 2006).

Another effect that we do not reproduce is the tie effect. The tie effect is the relative ease of problems of the form `N x N`, such as `3 x 3`, which can explain the preference for the answers `16` (`4 x 4`), and `9` (`3 x 3`) over `15` (`3 x 5`), which is the main discrepancy between the model and the data. Lebiere (1999) did match this effect, by using spreading activation in the model. The information on the screen and the information in working memory spread activation to the chunks in declarative memory. In the case of a tie-problem, twice as much activation is spread to each slot in declarative memory, making it easier to retrieve these facts. The current model does not include spreading activation, and therefore does not account for the tie-effect.

Together, the absence of spreading activation and not taking into account the frequency of exposure can explain the strange peak we find in the model data for the problem `3 x 4`. While the first four mistakes are relatively well matched, the answer `15` is overrepresented in the model data. This is a side effect from the way the similarity between numbers has been implemented. A number is more similar to the number that follows it than to the number that

precedes it. For example, the similarity of the numbers 3 and 4 is $-.25$ (3/4 − 1 = −.25), while the similarity between the numbers 4 and 5 is $-.2$ (4/5 − 1 = −.2). This would be canceled out by the higher activation of chunks that are more practiced (smaller problems) and that are easier to retrieve (tie-problems). Both problems will be resolved in a future version of the model.

6 x 9

The pattern of errors for the problem 6 x 9 deviates significantly from the data. One reason for this is that the current model relies quite heavily on the retrieval strategy, while children probably also use an algorithmic strategy (van der Ven et al., 2015). The errors the model makes are therefore retrieval related errors, while the errors in the data seem to be caused by mistakes in repeated addition.

One possible explanation for this phenomenon is the number of times a student is exposed to a problem. As mentioned above, smaller problems, such as 1 x 2 and 3 x 4, are more prevalent in schoolbooks, but also outside of school. While the current model assumes the same frequency of exposure and a similar learning curve for both easy and hard problems, this may not necessarily be the case.

Discussion

In this paper we have presented a basic model for single digit multiplication. We have implemented and extended the model by Lebiere (1999) in the PRIMs cognitive architecture. Our goal is to explain the error patterns found in the Math Garden data. With our model we have shown that we can fit some of the most common errors found in the data. Furthermore, the model also seems to capture the learning process from making mistakes to achieving the correct responses, since it starts out giving incorrect answers, but gradually gives more correct responses. However, the current model has difficulties explaining the data of the harder problems, which seem to be solved using different strategies than easier problems.

Because the amount of data from Math Garden, we assume that the errors are not specific to this dataset. When we compare the data from Math Garden to, for example, the data from Siegler (1988), there seem to be discrepancies in the kinds of errors that are made and in the relative frequency of the errors. However, because the data in the study by Siegler only comprises a small subset of students of a small age group, we believe that the math garden dataset is better suited to use as a basis for modeling errors.

While the model is not yet complete in the sense that it captures all of the effects found in the literature, it can capture the error patterns that children make while doing easy single digit multiplication problems and is therefore a first step in the understanding of the strategies and misconceptions that lead to mistakes. Our goal is to build a model that can fully explain our current data with regard to errors and correct responses, and which is able to predict the outcomes on new problems accordingly.

The model we present here is still work in progress. Future endeavors will focus on incorporating the effects found in the data, such as the problem size effect and the tie effect. As suggested above, the model will benefit from the implementation of spreading activation. Furthermore, instead of a model of specific problems, the goal is to build a model that is exposed to the full range of multiplication problems will give a better indication of the relative importance of specific multiplication facts in memory.

Another goal is to show the relationship between the multiplication skill and other skills that are taught at school, such as arithmetic skills. The choice of the cognitive architecture is therefore not a coincidence: the PRIMs architecture is specifically developed to be able to systematically investigate interactions and relationships between different tasks. By investigating the relationship between different tasks, we hope to elucidate the existence of different strategies that are used to solve relatively simple tasks.

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